- Consider the N-periodic sequence x with Fourier series coefficient sequence a.
- If x is real, then its Fourier series can be rewritten in trigonometric form as shown below.
- The trigonometric form of a Fourier series has the appearance

$$x(n) = \begin{cases} \alpha_0 + \displaystyle\sum_{k=1}^{N/2-1} \left[\alpha_k \cos\left(\frac{2\pi kn}{N}\right) + \beta_k \sin\left(\frac{2\pi kn}{N}\right) \right] + \alpha_{N/2} \cos \pi n & N \text{ even} \\ \alpha_0 + \displaystyle\sum_{k=1}^{(N-1)/2} \left[\alpha_k \cos\left(\frac{2\pi kn}{N}\right) + \beta_k \sin\left(\frac{2\pi kn}{N}\right) \right] & N \text{ odd,} \end{cases}$$

where $\alpha_0 = a_0$, $\alpha_{N/2} = a_{N/2}$, $\alpha_k = 2 \operatorname{Re} a_k$, and $\beta_k = -2 \operatorname{Im} a_k$. Note

- that the above trigonometric form contains only *real* quantities.

- Letting $a'_k = Na_k$, we can rewrite the Fourier series synthesis and analysis equations, respectively, as

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} a'_k e^{j(2\pi/N)kn} \quad \text{and} \quad a'_k = \sum_{n=0}^{N-1} x(n) e^{-j(2\pi/N)kn}.$$

- Since x and a' are both N-periodic, each of these sequences is completely characterized by its N samples over a single period.
- If we only consider the behavior of x and a' over a single period, this leads to the equations

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} a'_k e^{j(2\pi/N)kn} \quad \text{for } n = 0, 1, \ldots, N-1 \quad \text{and}$$

$$a'_k = \sum_{n=0}^{N-1} x(n) e^{-j(2\pi/N)kn} \quad \text{for } k = 0, 1, \ldots, N-1.$$

- As it turns out, the above two equations define what is known as the discrete Fourier transform (DFT).

- The discrete Fourier transform (DFT) X of the sequence x is defined as

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j(2\pi/N)kn} \quad \text{for } k = 0, 1, \ldots N-1.$$

- The preceding equation is known as the DFT analysis equation.
- The inverse DFT x of the sequence X is given by

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j(2\pi/N)kn} \quad \text{for } n = 0, 1, \ldots N-1.$$

- The preceding equation is known as the DFT synthesis equation.
- The DFT maps a finite-length sequence of N samples to another finite-length sequence of N samples.
- The DFT will be considered in more detail later.

- •Since the analysis and synthesis equations for (DT) Fourier series involve only *finite* sums (as opposed to infinite series), convergence is not a significant issue of concern.
- •If an N-periodic sequence is bounded (i.e., is finite in value), its Fourier series coefficient sequence will exist and be bounded and the Fourier series analysis and synthesis equations must converge.

Section 9.2

Properties of Fourier Series

$$x(n) \xleftrightarrow{\text{DTFS}} a_k \quad \text{and} \quad y(n) \xleftrightarrow{\text{DTFS}} b_k$$

Property	Time Domain	Fourier Domain
Linearity	$\alpha x(n)+\beta y(n)$	$\alpha a_k+\beta b_k$
Translation	$x(n-n_0)$	$e^{-jk(2\pi/N)n_0} a_k$
Modulation	$e^{j(2\pi/N)k_0 n} x(n)$	a_{k-k_0}
Reflection	$x(-n)$	a_{-k}
Conjugation	$x^*(n)$	a^*_{-k}
Duality	a_n	$\frac{1}{N}x(-k)$
Circular convolution	$x \circledast y(n)$	$N a_k b_k$
Multiplication	$x(n)y(n)$	$a \circledast b_k$
Even symmetry	x even	a even
Odd symmetry	x odd	a odd
Real	$x(n)$ real	$a_k = a^*_{-k}$

Property	
Parseval's relation	$\frac{1}{N}\sum_{n=\langle N\rangle} \lvert x(n)\rvert^2 = \sum_{k=\langle N\rangle} \lvert a_k\rvert^2$

- Let x and y be N-periodic signals. If $x(n) \stackrel{\text{DTFS}}{\longleftrightarrow} a_k$ and $y(n) \stackrel{\text{DTFS}}{\longleftrightarrow} b_k$, then

$$\alpha x(n) + \beta y(n) \stackrel{\text{DTFS}}{\longleftrightarrow} \alpha a_k + \beta b_k,$$

  where α and β are complex constants.
- That is, a linear combination of signals produces the same linear combination of their Fourier series coefficients.

- For an N-periodic sequence x with Fourier-series coefficient sequence a, the following properties hold:

$$x \text{ is even} \Leftrightarrow a \text{ is even}; \quad \text{and}$$
$$x \text{ is odd} \Leftrightarrow a \text{ is odd}.$$

- In other words, the even/odd symmetry properties of x and a always match.

- A signal x is *real* if and only if its Fourier series coefficient sequence a satisfies

$$a_k = a^*_{-k} \text{ for all } k$$

)i.e., a has *conjugate symmetry*.(

- From properties of complex numbers, one can show that $a_k = a^*_{-k}$ is equivalent to

$$|a_k| = |a_{-k}| \quad \text{and} \quad \arg a_k = -\arg a_{-k}$$

)i.e., $|a_k|$ is *even* and $\arg a_k$ is *odd*.(

- Note that x being real does *not* necessarily imply that a is real.

- For an N-periodic sequence x with Fourier-series coefficient sequence a, the following properties hold:
  1. a_0 is the average value of x over a single period;
  2. x is real and even $\Leftrightarrow a$ is real and even; and
  3. x is real and odd $\Leftrightarrow a$ is purely imaginary and odd.

Section 9.3

Fourier Series and Frequency Spectra

- The Fourier series provides us with an entirely new way to view signals.
- Instead of viewing a signal as having information distributed with respect to *time* (i.e., a function whose domain is time), we view a signal as having information distributed with respect to *frequency* (i.e., a function whose domain is frequency.(
- This so called frequency-domain perspective is of fundamental importance in engineering.
- Many engineering problems can be solved *much more easily* using the frequency domain than the time domain.
- The Fourier series coefficients of a signal x provide a means to *quantify* how much information x has at different frequencies.
- The distribution of information in a signal over different frequencies is referred to as the *frequency spectrum* of the signal.

- To gain further insight into the role played by the Fourier series coefficients a_k in the context of the frequency spectrum of the N-periodic signal x, it is helpful to write the Fourier series with the a_k expressed in *polar form* as

$$x(n) = \sum_{k=0}^{N-1} a_k e^{j\Omega_0 kn} = \sum_{k=0}^{N-1} |a_k| e^{j(\Omega_0 kn + \arg a_k)},$$

  where $\Omega_0 = \frac{2\pi}{N}$.
- Clearly, the kth term in the summation corresponds to a complex sinusoid with fundamental frequency $k\Omega_0$ that has been *amplitude scaled* by a factor of $|a_k|$ and *time-shifted* by an amount that depends on $\arg a_k$.
- For a given k, the *larger* $|a_k|$ is, the larger is the amplitude of its corresponding complex sinusoid $e^{jk\Omega_0 n}$, and therefore the *larger the contribution* the kth term (which is associated with frequency $k\Omega_0$) will make to the overall summation.
- In this way, we can use $|a_k|$ as a *measure* of how much information a signal x has at the frequency $k\Omega_0$.

- The Fourier series coefficients a_k of the sequence x are referred to as the frequency spectrum of x.
- The magnitudes $|a_k|$ of the Fourier series coefficients a_k are referred to as the magnitude spectrum of x.
- The arguments $\arg a_k$ of the Fourier series coefficients a_k are referred to as the phase spectrum of x.
- The frequency spectrum a_k of an N-periodic signal is N-periodic in the coefficient index k and 2π-periodic in the frequency $\Omega = k\Omega_0$.
- The range of frequencies between $-\pi$ and π are referred to as the baseband.
- Often, the spectrum of a signal is plotted against frequency $\Omega = k\Omega_0$ (over the single 2π period of the baseband) instead of the Fourier series coefficient index k.

- Since the Fourier series only has frequency components at integer multiples of the fundamental frequency, the frequency spectrum is *discrete* in the independent variable (i.e., frequency.(
- Due to the general appearance of frequency-spectrum plot (i.e., a number of vertical lines at various frequencies), we refer to such spectra as line spectra.